

Icosian Game. A Strategy of Configuration and Exploration of Musical Forms

1. Prolegomena

Motto: "Die Matematik ist nur das Mittel für die letzte und allgemeine Menschenerkenntniss"
 [Friedrich Nietzsche: Fröhliche Wissenschaft]

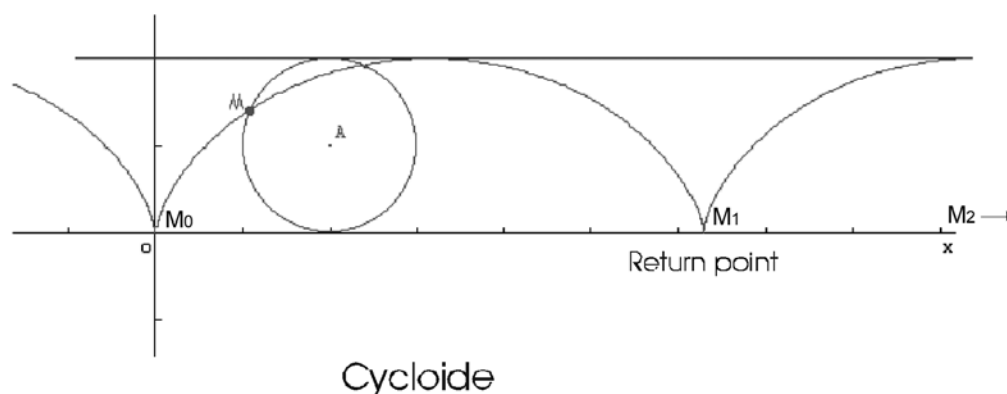
It is obvious from the beginning that the notion of cycle derives from the Greek *kyklos*. In any of today's modern languages, it uses the same root as that from which both the attribute *cyclic* and the deverbative *cyclicity* derive. As a musicological term in musical art, it defines a compositional phenomenon that is related to the writing traits and/or to certain constructive aspects determined both by the composer's preferences and by the coercive framework of the style in which the discourse is written. Various dictionaries, treaties and reference instruments define the phenomenon relatively differently, but, in general, are in accord with the aspects mentioned above.

We should, however, note that most of the theoretical sources consider that the term has a specific, compositional application and that the different aspects encountered in various musical discourses can be subsumed under one and the same fundamental principle: the cyclic principle. This view is at least limited if not reduced, and in some aspects, even false. To give the cyclic phenomenon a substantial and coherent definition we must resort to concepts that are at the same time simpler and stronger, able to substantiate a global and coherent definition.

The concept of cyclic phenomenon is a universally perceptible reality. Both in modern and in today's physics, most phenomena occur (at least in certain sections of the physical processes) under the form of cycles, returns, identical or varied reiterations. Most physical phenomena at micro- and macrostructural level display this type of cyclic phenomena, from the domain of chemical reactions, body dynamics, meteorology, astronomy, general astrophysics etc. However, these cyclic phenomena should be included within the more general context of an entropic global evolution which excludes cyclicity at the highest level. Only the extreme levels (the *sub quantum* and the astrophysical level of the *macro universe*) do not display cyclicities.

The very idea of cycle and cyclicity needs a clearer and more profound definition. The science that can give it an accurate, apodictic, definition is mathematics. First, Euclidian geometry, defining all the phenomena related to circle geometry which are, essentially, cyclic in nature, then mathematical analysis, defining certain curves as being cyclic, and lastly advanced mathematics, concerned with group theory, transformation groups, topology, ring theory etc.

Essentially, if every time we have in mind the idea of cycle we think of the figure of the circle and of the related corollary aspects, then in the arts of the discourse and, particularly, in the field of music, due to the oriented existence of the Time vector, the correct representation should always be the cycloid and the related family of curves.



On the other hand, in what concerns the phenomenology of the musical discourse, we find the same relative lack of consistency in the denomination and in the definition of the cyclic phenomena. This is due to the existence of an extremely wide area of application of these phenomena, which are different qualitatively and quantitatively, but for which we use the same notion or concept. Although the extreme cases of cyclicity: *cyclicity* = 0 (~ quasi-infinitely non-repetitive, chaotic discourse) and, respectively, *cyclicity* = 1 (~ which should be an absolutely repetitive discourse of a simple or complex section repeated quasi-infinitely) do exist in musical art, they are isolated and extremely rare. For the first case we can quote certain 20th century non-repetitive discourses, constructed deliberately in this direction, which permanently exploit new aspects of content and form. For the second case we can quote discourses which are undefined in time, such as the Tibetan incantations, or the strictly repetitive minimalist music. Although significant, these discourses do not generally have an artistic sense from a compositional viewpoint, they being rather mechanically-generated discourses. Expressively, however, they can reach a level of musical incantation that can become quite significant.

2. Subjectio

Motto: "Musik sei ihrem Wesen nach Algebra"
[Novalis]

Cycle, Cyclic and Cyclicity in the Musical Discourse

The first hypostasis of creating a compositional cycle is that in which the determining vector is not the musical, but the poetical one. In the case when music treats (as in the lied, for example) certain poetic texts that are grouped within a cycle, the final musical elaboration will be a cycle, too. It is the case of the lied cycle op. 24 (*Junge Leiden* – Heine), op. 39 (Eichendorff), op. 42 (*Frauenliebe und Leben* – Chamisso), or op. 53 (*Dichterliebe* – Heine) by R. Schumann. In this case, music borrows a ready-made structure and generates what in the title is entered as: Cycle. Liederkreis etc. This genre is called cycle derived from the poetic primary source.

The second case is more interesting, as on form level it achieves cyclic structures that are derived from the poetic structure and finds analogue solutions in the musical language. The solution dates back to the times of the Greek lyrical music (the hymn or the strophic ode), while other interesting and significant solutions lie in the music of the troubadours (the ballad, the virelai etc.). What is relevant is the construction of a musical section (chorus, envoi) which recurs periodically. The constructive solution was then transferred to certain instrumental forms (rondo, ritornel) and became purely musical. Thus, a principle of cyclic construction affirms itself for the first time in a strictly musical context.

The cyclic phenomenon can manifest itself at various levels of musical structure, from that of the microstructure at cellular, figural, motivic, phraseological, periode or form level, to that of the integrality of the musical genre. However, we believe that these types of cyclicity are hierarchically different and therefore should also be defined by means of specific notions, as they describe varied and nonequivalent phenomena. What does cyclic actually mean in the musical discourse? An a priori definition requires the capacity to distinguish among the various sections and to attribute to some, a higher constructive sense through their intrinsic capacity (or maybe only through the composer's will?) to generate, by way of reiteration, new significant structures and to substantiate a longer, more profound discourse.

Such an issue obviously integrates itself within the essential distinction between the fundamental opposition between Identity – Alterity. What exactly entitles us to speak of the "unity" of a certain musical conglomerate and of its capacity to substantiate larger units by way of reiteration and of varied compositional procedures? This in fact is the essential issue of describing the musical discourse in the light of the musical forms theory which acts by opposing the identity of the material and the alterity of certain contrasting parts. How much identity should there exist at the level of a discourse deemed cyclic? How much alterity should there exist for the recurrence or reiteration to be different and significant?

If we take this criterion as valid, then all the "symmetrical" forms of the type ABA, ABCBA, ABCDCBA are cyclic, as they describe a cycloid which has the characteristic traits of an evolutive cycle subsequently closed. These well-known forms of the musical discourse have been explored

for centuries, but were used at a higher level in the construction of the musical discourse in the music of the 20th century. Such symmetric solutions were used by Anton Webern or Béla Bartók and later by Romanian composers like Aurel Stroe in "Arcade" (Arcads) or Cornel Țăranu in "Ghirlande" (Guirlands). The titles are programmatic and significant, for the music itself is conceived on cycloid trajects.

However, on a more profound level we should take into account the possibility of construction in terms of treating an initial material in varied forms and discourses at syntax, dimension, specific place and concrete approach level. Thus, any variational discourse should be defined by excellence as cyclic, since in each variation it repeats the initial material in other forms, structures and expressions. Among the examples of highest perfection of this type of cyclicity, we should probably define "The Goldberg Variations", beginning and ending with the thematic "Aria", but which can also be grouped in larger periodicities of 10x3, 6x5, 5x6 or 3x10, according to the meaning we assign to the cyclicity of the variations and analytical hypotheses that we are willing to approach. The variational cycle holds an important place in Anton Webern's work. Mention should be made here of the Variations Op. 27, or of the 2nd part of the Symphony Op. 21 – Variations. These have the symmetric structure A-B-C-D-Cv-Bv-Av, where in respect to the symmetry axis the variation is the 4th.¹

Another hypostasis of cyclicity occurs in composite genres like sonatas, suites and baroque partitas, which often make use of common intonational melodic typologies to generate certain extremely different parts in terms of several parameters (tempo, meter, dynamics, character etc.). Practically, this can be found just as well in certain old musics, such as the technique of constructing the Middle East musical discourses on an essential makkam, or the use of specific figures in elaborating the Gregorian chant, or generally wherever we have a musical style whose writing techniques and stylistic features hierarchically impose certain international typologies as fundamental. Wherever there is a style, there are "stylemas". These stylemas can be tetrachordal for the vocal music of the Renaissance, the tetrachord or hexachord for the music of the Baroque, the trichord or trisone for the Classicism, more complex melodic harmonic typologies for the Romanticism etc., up to the constructive typologies of the dodecaphonic writing. These substantiate the construction of "cyclic nuclei" used subsequently in the compositional strategy.

Probably the most profound meaning and the most important compositional significance of the concept of cyclic, cyclicity refers to the capacity of certain microstructures to enable the composer to impose unity and coherence to a large-dimensioned composite structure. In this sense, the cyclic phenomenon has already imposed itself upon the superior conscience of certain great composers, evolving from possibility to necessity and thence to the compulsoriness of such a complex constructive solution. Haydn's or Mozart's chamber and symphonic music displays striking, though discreetly and parsimoniously used examples. Beethoven is considered to have been the one who introduced this composition technique as a necessity and who, thanks to the importance and complexity of his compositional thinking, "imposed" its use as an quasi-obligatory requirement for any subsequent composition attempt. He is looked upon as a "catastrophic" phenomenon in the history of the European compositional thinking, an incommensurable leap in the compositional conception and will, and while no music could be written thereafter without either following his path or being against it, he could be, in no case, no longer ignored.

Here was a rather brief case history of the variability of this concept, as well as the extremely different composition solutions it can produce. If during the past centuries the musical discourse used to be achieved within a relatively coherent stylistic and syntactic framework and its techniques could be defined even in the absence of the composer's comments, the 20th century music poses more difficult, more specific and at times even insurmountable challenges in the handling of this concept. We are in a completely different historical, mental and cultural context. The concept of general style disappears and is divided in just as many styles as there are great composers. Scriabin, Bartók, Stravinski, Schönberg, Berg, Messiaen, Schostakovich, Prokofiev, Boulez, Stockhausen, Lutosławski represent just as many different styles and languages. The composer's quasi-obligation to be original on a strictly individual level and to search for new and unexplored techniques of elaboration of the musical discourse becomes general and inherent as an ontological condition of creation. A new and imperious necessity is imposed on the composer to become himself a theoretician and musicologist in the explanation and analysis of his own language.

Thus, even though the technique in itself and the idea of cyclicity do exist, they are conceived, achieved and received in a fundamentally different way. All of these are supplemented by an ever greater, sometimes insurmountable difficulty in the real-time perception of the composer's intentions and of the writing elements that are objectively present in the score. All the above put under a question mark the validity of the technique itself, which is no longer perceived as an effort to confer unity, coherence and integrity to a discourse that is diverse in time, it being decodable only upon a very professional, detailed and attentive analysis. In the end, we can ask ourselves if we do have a good definition for the sense of cyclic and for the cyclic techniques applied in the musical writing. Which should the identity level of the sonorous material be, to justify the qualifier of cyclic? How far can the alteration procedures be pushed (variation, handling, development, cryptic essentialization) in order to constitute a reference plan for the idea and qualifier of cyclic? And, after all, there are objective characteristics of the musical expressivity that can emphasize the cyclic character of a work, which in most cases is mental and constructive in essence.

All this procedures are in fact particular cases of an general and fertile phenomenon – the Symmetry.

3. Adjectio

Motto: "Der Musik kommt die Harmonie zu, wie die Symmetrie der Mathematik"
[Heinrich von Kleist. Brief an Marie von Kleist. 1811]

Musical forms and mathematical isomorphism

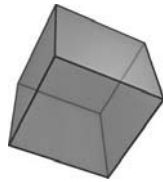
In the evolution of human culture, considered as a whole (meaning the integration of artistic creation, the rational-scientific and abstract mathematic) certain fundamental concepts have emerged as essential models of the formal configuration. These models – true paradigms of thinking – have a general capacity of application in any zone of thinking and/or creation. One of these universal paradigms is the geometric spatial theory of the regular polyhedral solids.

Discovered in the Greek antiquity (apparently even before the theory of polyhedral solids established by the Pythagorean school) and intensely studied and interpreted on the gnoseological and metaphysical levels, these solids become essential models of the Greek thinking. The Pythagoreans postulated these regular polyhedra (the tetrahedron, cube or hexahedron, octahedron, and icosahedron) at the basis of existence as fundamental structures of essential elements, respectively, fire, earth, water and air. It is an ancient metaphysical conception coming from the Ionian philosophy and developed in the School of Crotona. The last and most complex one, the dodecahedron (the only based by the connection of the regular pentagon in dihedral angles, that implies the golden number – ϕ^2), was identified as The Spirit and/or The Universe in its totality. Because the notions of this theory have been apprehended in the platonic dialog "Timaeus", exposing the most complexly and rationally articulated cosmological conception of the Greek antiquity, the polyhedra became known under the denomination of "platonic solids".

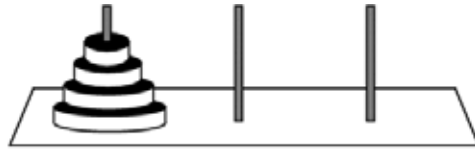
The intrinsic beauty and the amazing mathematical proprieties of these five forms have fascinated the scholars (philosophers, mathematicians, artists) from Plato to NATO. The complete analysis of platonic solids, given in the last book of Euclid's *Elements* is the coronation of this singular work. Johannes Kepler believed his entire life in the correspondence of platonic solids and the six planetary orbits (known in his time) by the arrangement of the five regular solids inside the Saturn orbit. Moreover, he conceived musical harmony in the strict letter of the theory of these regular solids. The complete topologic theory of the polyhedra was established by Euler in the 18th century. Today mathematicians have ceased to regard the polyhedra with such a mystical respect, but they have continued to study the theme in the frame of topology or theory of graphs or groups. We have asserted these concise historical considerations in order to emphasize the extraordinary influence of these platonic solids over the entire human knowledge, and mostly because even today, there are sources of fascination and revelatory surprises.

"There are few experiences so exciting for the mathematician like the discovery that in fact two mathematical structures apparently independent from each other, are in reality in close connection".³ This is the phenomena called isomorphism. In the 19th century, the great British mathematician William Hamilton invented a special game named "icosian game", based on the continuous graph along the edges of solid angles of polyhedra. Not long after that, William Crowe

discovered an amazing congruence between the icosian game and another game named "The Hanoi Tower" (a very old oriental game with mystical connotations) has been reinvented in Europe by E. Lucas in 1883. To be more exact, between the hexahedron and the Hanoi tower the strategies of complete exploration (cube) and complete mutation of elements (of the tower) are strictly isomorphs, i.e. identical.



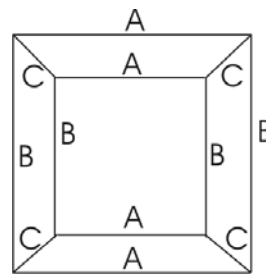
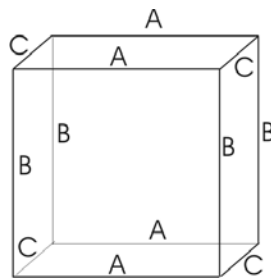
Cube (hexahedron)



Hanoi Tower

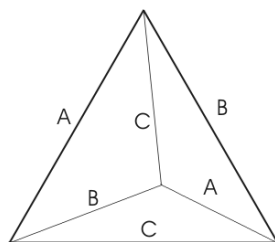
We don't intend to explore the details of geometrical or philosophical development of this phenomenon, but we notice a very significant congruence between these two "games" and the "game" of musical forms as strategy of placing different parts in the musical discourse. The specific algorithm for the covering of all the vertices of the cube (8 vertices, 12 edges, 6 faces) along the edges (with the restrictive condition: never pass twice by the same vertex) and the "resolution" of Hanoi Tower (equally restricted by the condition: never have a bigger disk upon a small one) gives the following series of elements: ABACABA. In this series the musician can intuitively recognize the well-known model of the musical forms: the classical rondo. It goes without saying that we must have a notation of each edge of the hexahedron that connects two vertices (the edges oriented in the same direction are similarly notated: A, B, or C).

The graph in two dimensions is:

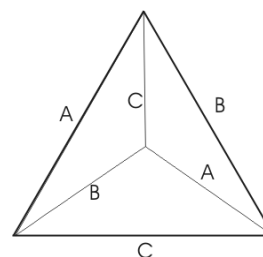


However, we accomplish the task, we will obtain the same formula: ABACABA or ACABACA, which are similar or isomorph. The algorithm is strictly determined, no matter what the point of beginning or the selection of edges is.

If we explore other politopes beginning from one vertex we will obtain other formulae strictly determined by the number of vertices and the relation established by the edges. In the case of tetrahedron (6 vertices, 6 edges, 6 faces) no matter of the vertice of beginning or the direction (with the single condition: to be a hamiltonian graph) we will obtain always one formula ABA which describes the tripartite form with reprise:



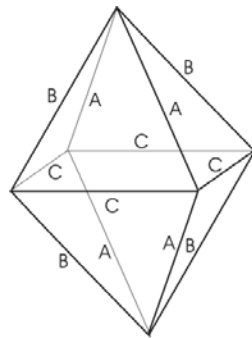
Tetrahedron



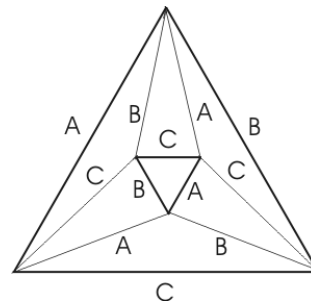
Tetrahedron 2-dimensional graph

In this case the formula ABA is equivalent of ACA or BAB, BCB, CAC, CBC.

Going further in our examination considering the octahedron (6 vertices, 12 edges, and 8 faces) we will obtain a new formula than correspond at this particular structure. In fact the octahedron is symmetrically isomorphic to the cube and the resultant formula is ABCAB.



octahedron



octahedron – 2-dimension graph

We can start the graph anywhere and the results will be also:

AB C AB	BA C BA	CA B CA
AC B AC	BC A BC	CB A CB

all identical because the different formulae results by the symmetry of rotation. This is the formula of the bitematic sonata.

Considering the history and evolution of musical forms it is very clear that they are not elaborated by a single composer or in a single musical style. They are constructed slowly and painfully by intellectual effort by a sort of collective finding the best solution at the problematic of composition. It is the result of a discursive process cumulating lent contributions and continuous optimizations. By this process results the musical forms as known in the classical theory of forms. It is no trace, not a clue to indicate a conscientious realization of forms as polyhedral graphs, therefore it is excluded the possibility of an intentional connection. But the reality is here supported by strong arguments and the complete isomorphism of the three cases presented is not only exciting or disturbing, but also generator of new and important significations.

4. Postlegomena

Motto: "Musikalische Form ist Vorsprung der Macht in der Raum"
[Hans Mersmann. Formenlehre]

The "Hanoi tower" is known in the ancient Asia in different hypostasis, the most notorious one being legendarily located in the Temple of Benares (in India). By this legend, the priests of the temple were always engaged in a continuous work consisting in the complete mutation of 64 disks carved in crystal. Once the transfer accomplished, the exact moment of the end of the world will be attained and the Universe will collapse.⁴ It is obviously an ancestral model of action and thinking, corresponding in spirit to an archetypal and paradigmatic Buddhist image of the Universe.

Likewise, the polyhedra were known in the Asiatic world and in the Mediterranean space long before their theorization in the Pythagorean School or before the Platonic commentaries. They correspond to a fundamental image of the structure of the world, as superior order and symmetry.

The musical forms, elaborated in a long process of discursive evolution and studied by the historic and systematic musicology, obviously evoke the same superior models of order and symmetry. It seems very likely that the human creative spirit, irrespective of the nature of the problems it has to face, has the same algorithms of exploration of the figurative reality and of configuration of spiritual order.

"In the same Platonic manner, the final theory of the Universe will be characterized by a series of important exigencies of symmetry, which we can already indicate today. Even if we examine this theory with the whole skepticism – which defines the scientist's highest obligation, – we are allowed to affirm that we have reached structures of extraordinary simplicity, unity and beauty and that these structures appear to be more important because they are true not only in a special field of physics, but for the whole Universe"¹⁵.

In the light of the newest theory of physics (*theory of whole* and especially *theory of chaos*) it seems that a chaotic phenomenon (large classes of phenomena are chaotic) is in fact determined by subtle laws of equilibrium: the so-called "strange attractors". The construction of musical forms could be described as a chaotic phenomenon. The creation of hundreds of composers tends to be regulated in this case this by a strange attractor: the regular polyhedra.

And if in the model of Platonic solids, or in a game with archetypal mystical correspondences, it is possible to find the unity of the whole world according to Heisenberg's assertions, it is very probable that in their essence the musical forms reflect the same superior order and unity. Apparently separated and historically disseminated in the time and space of the human culture, the evolution of musical forms seems to have been determined by higher (or more profound) laws of symmetry and structure.

These laws are situated at a higher level, above epochs, styles and individual creation. They are supra-stylistic hypostasis and each period or style will repeat and re-find the same structures. In fact, the most "troubling" coincidence in this case is the superposing of a temporal configuration, in a perfect congruence with the spatial one. Hans Mersmann has seen it right: "the musical form is the projection of the force in the space".

Abstract

The paper is divided in four sections:

1. *Prolegomena*. The concept of cyclic phenomenon is a universally perceptible reality

The very idea of cycle and cyclicity needs a clearer and more profound definition. The science that can give it an accurate, apodictic, definition is mathematics. Particularly, in the field of music, due to the oriented existence of the Time vector, the correct representation should always be not the cycle but cycloid and the related family of curves. We find relative lack of consistency in the denomination and in the definition of the cyclic phenomena. This is due to the existence of an extremely wide area of application of these phenomena, which are different qualitatively and quantitatively, but for which we use the same notion or concept.

2. *Subjectio*. Cycle, Cyclic and Cyclicity in the Musical Discourse

There are examined different hypostasis of the phenomenon and identified the sources, also the specific principles of construction. Such an issue obviously integrates itself within the essential distinction between the fundamental opposition between Identity – Alterity. This in fact is the essential issue of describing the musical discourse in the light of the musical forms theory which acts by opposing the identity of the material and the alterity of certain contrasting parts. How much identity should there exist at the level of a discourse deemed cyclic? How much alterity should there exist for the recurrence or reiteration to be different and significant? Thus, even though the technique in it and the idea of cyclicity do exist, they are conceived, achieved and received in a fundamentally different way. In the end, we can ask ourselves if we do have a good definition for the sense of cyclic and for the cyclic techniques applied in the musical writing.

3. *Adjectio*. Musical forms and mathematical isomorphism

There are asserted some concise historical considerations about polyhedra, in order to emphasize the extraordinary influence of platonic solids over the entire human knowledge, and mostly because even today, there are sources of fascination and revelatory surprises. We find a significative congruence between Hamiltonian graphs on polyhedra and important formal solution in the field of musical forms like: tripartite form with reprise, rondo, and bitematic sonata. Considering the history and evolution of musical forms it is very clear that they are not elaborated by a single composer or in a single musical style. They are constructed slowly and painfully by intellectual effort by a sort of collective finding the best solution at the problematic of composition. It is no trace, not a clue to indicate a conscientious realization of forms as polyhedral graphs, therefore it is excluded the possibility of an intentional connection. But the reality is here supported by strong arguments and the complete isomorphism of the three cases presented is not only exciting or disturbing, but also generator of new and important significations.

4. *Postlegomena*

Like the polyhedra or the Hanoi tower, the musical forms, elaborated in a long process of discursive evolution and studied by the historic and systematic musicology, obviously evoke the same superior models of order and symmetry. It seems very likely that the human creative spirit,

irrespective of the nature of the problems it has to face, has the same algorithms of exploration of the figurative reality and of configuration of spiritual order. The construction of musical forms could be described in the light of Chaos theory as a chaotic phenomenon. The creation of hundreds of composers tends to be regulated in this case this by a strange attractor: the regular polyhedra.

Notes

- ¹ Webern himself says: "the forth variation it is the midpoint of the whole movement, after which everything goes backward", quoted after Robert U. Nelson: *Webern path to the serial variation*, PNM, Vol. 7, No. 2 (1969), pp. 72–79.
- ² In fact ϕ as irrational number, are the most irrational number known today in mathematics.
- ³ Gardner, Martin. *Mathematics, Magic and Mystery*, Dover, 1956, p. 73.
- ⁴ This "cosmic" rondo ABACADAE EADACABA (assuming one movement per each second) will be accomplished in approximately 584, 942, 417, 355 years!
- ⁵ Heisenberg, Werner. *Descoperirea lui Plank și problemele filosofice fundamentale ale teoriei atomului* in: W. Heisenberg, "Pași peste granițe", Ed. Politică, București, 1977, p. 34. [German edition: *Schritten über Grenzen*, Frankfurt, 1967]

Santrauka

***Icosian Calculus* žaidimas. Muzikos formų konfigūravimo ir tyrinėjimo strategija**

Muzika ir matematika turi daug panašumų ir galimų sąsajų. Muzikos formą neabejotinai galima nagrinėti pagal daugybę matematikos teorijų, tokių kaip kombinatorinė algebra, mozaikos teorija, ribų teorija ir kt.

Pagrindinio šioje konferencijoje nagrinėjamo klausimo (ciklo idėja muzikoje) kontekste verta dėmesio yra airių matematiko W. R. Hamiltono (1805–1865), įvedusio Hamiltono ciklo sąvoką, teorija.

Hamiltono ciklas – tai toks ciklas, kuris eina per kiekvieną briaunainio viršūnę (bet kelių dimensijų erdvėje) tik vieną kartą. Taikant Hamiltono ciklą muzikos formai, jis yra griežtai homomorfiškas. Visi briaunainio grafai veikia kaip modeliai, tiek sprendžiant pagrindinius formos klausimus, tiek ir organizuojant teminę medžiagą, pvz., ABA forma, rondo forma, sonatos forma ir kt.

Net jei muzikinis mąstymas ir neturi tvirto matematinio pagrindo, griežtas homomorfizmas gali padėti išspręsti, regis, visiškai skirtingas problemas. Ir, žinoma, abiejose srityse veikia ciklo fenomenas.